Plasma Physics – II

21-02-2019. Thursday.

Hydromagnetic waves:-

We shall take into acount both electrons and ions.

i) Alfven wave.

ii) Magnetosonic wave.

Alfven wave:-

low frequency waves. It means

$$\omega << \omega_{ce} = \frac{eB_0}{m_e}$$

$$\omega << \omega_{ci} = \frac{eB_0}{m_i}$$

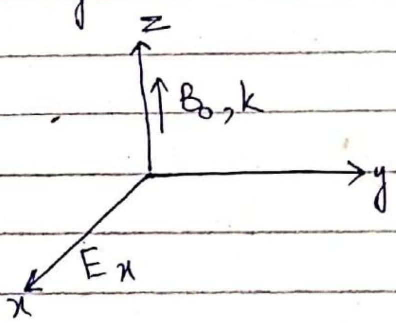
→ Ions are low frequency waves because their inertia is high.

Suppose that Electric field is along x-axis, magnetic field is along z-axis & $\vec{k}$ is along $\hat{z}$ axis.

$$\bar{E} = E_x\,\hat{x}$$

$$\bar{B} = B_0\,\hat{z}$$

$$\bar{k} = k\,\hat{z}$$



Maxwell's Equation:

$$\bar{\nabla}\times\bar{B} = \frac{4\pi}{c^2}\bar{J} + \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t} \quad \text{—(1)}$$

This is electromagnetic wave So, $\bar{B}$, E are perturbed

$$\bar{\nabla}\times\bar{E}_1 = -\dot{\bar{B}}_1 \quad \text{—(2)}$$

Differentiating eq.(1) w.r.t "time".

$$\bar{\nabla}\times\dot{\bar{B}}_1 = \frac{4\pi}{c^2}\dot{J}_1 + \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{—(3)}$$

From eq.(2) value of B_1 substiting in eq.(3)

$$-c^2[\bar{\nabla}\times(\bar{\nabla}\times\bar{E}_1)] = \underline{4\pi}\,\dot{J}_1 + \underline{1}\,\ddot{E}_1$$

$$-c^2[\bar{\nabla}\cdot\bar{E}_1 - \nabla^2 E_1] = \frac{4\pi}{c^2}\dot{J}_1 + \frac{1}{c^2}\ddot{E}_1$$

$\bar{\nabla}\cdot E_1 = 0$ because E & k are perpendicular to each other in transverse wave.

$$c^2\nabla^2 E_1 = \frac{4\pi}{c^2}\dot{J}_1 + \frac{1}{c^2}\ddot{E}_1$$

$$c^2(-k^2E_1) = \frac{4\pi}{c^2}(-i\omega J_1) + \frac{1}{c^2}(-\omega^2)E_1$$

$$(\omega^2 - c^2k^2)E_1 = -4\pi i\omega J_1$$

Writing the x-component of above equation,

$$(\omega^2 - c^2k^2)E_x = -i4\pi\omega n_0 e(V_{i1} - V_{e1}) \qquad \because J = ne(V_i - V_e).$$

$$(\omega^2 - c^2k^2)E_x = -i4\pi\omega n_0 e(V_{ix_1} - V_{ex_1}) \quad \text{—— (4)}$$

For the velocity we have to write the equation of motion,

Linearized form of Equation of motion for ions:

$$m_i \cancel{n_0}\frac{\partial V_{i1}}{\partial t} = \cancel{n_0} e[E_1 + V_{i1}\times B_0].$$

X-component:

$$m_i\frac{\partial V_{ix}}{\partial t} = e[E_{1x} + V_{iy}B_0]$$

By applying fourier transformation, we get

$$-i\omega V_{ix} = \frac{e}{m_i}(E_{1x} + V_{iy}B_0)$$

$$V_{ix} = +\frac{ie}{m_i\omega}[E_{1x} + V_{iy}B_0] \quad \text{—— (5)}$$

Y-component:

$$V_{iy} = \frac{ie}{m_i\omega}(0 - V_{ix}B_0)$$

$$V_{iy} = \frac{-ieB_0V_{ix}}{m_i\omega}$$

$$V_{iy} = -\frac{i\omega_{ci}}{\omega}V_{ix} \quad \text{—— (6)}$$

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Substituting eq. (6) into eq. (5)

$$V_{ix} = \frac{ie}{m_i\omega}\left(E_{1x} + \left(-\frac{i\,\omega_{ci}}{\omega}V_{ix}\right)B_0\right)$$

$$V_{ix} = \frac{ie}{m_i\omega}E_{1x} + \frac{\omega_{ci}\,e}{m_i\omega^2}V_{ix}B_0 \qquad \because \frac{eB_0}{m_i} = \omega_{ci}$$

$$V_{ix} = \frac{ie}{m_i\omega}E_{1x} + \frac{\omega_{ci}^2}{\omega^2}V_{ix}$$

$$\left(1 - \frac{\omega_{ci}^2}{\omega^2}\right)V_{ix} = \frac{ie}{m_i\omega}E_{1x}$$

$$V_{ix} = \frac{ie}{m_i\omega}E_x\left(1 - \frac{\omega_{ci}^2}{\omega^2}\right)^{-1} \quad \text{———} (7)$$

Similarly for the electron,

$$m_e \cancel{n_0}\frac{\partial V_{e1}}{\partial t} = -\cancel{n_{e0}}e\left(E_1 + V_{e1}\times B_0\right).$$

x-component:

$$m_e\frac{\partial V_{ex}}{\partial t} = -e\left(E_x + V_{ey}B_0\right)$$

$$-i\omega V_{ex} = \frac{-e}{m_e}\left(E_x + V_{ey}B_0\right)$$

$$V_{ex} = \frac{-ie}{m_e\omega}\left(E_x + V_{ey}B_0\right) \quad \text{———} (A)$$

Y-component:

$$V_{ey} = \frac{-ie}{m_e\omega}\left(0 - V_{ex}B_0\right)$$

$$V_{ey} = \frac{ie}{m_e\omega}V_{ex}B_0 \Rightarrow V_{ey} = \frac{i\omega_{ce}}{\omega}V_{ex} \quad \text{———} (B)$$

Put eq. (B) in eq. (A)

$$V_{ex} = \frac{-ie}{m_e\omega}\left(E_x + \frac{i\omega_{ce}}{\omega}V_{ex}B_0\right)$$

$$V_{ex} = \frac{-ie}{m_e\omega}E_x + \frac{\omega_{ce}^2}{\omega^2}V_{ex}$$

$$\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)V_{ex} = \frac{-ie}{m_e\omega}E_x$$

$$V_{e_x} = \frac{-ie}{m_e\omega}E_x\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)^{-1} \quad \text{—— (8)}$$

Substituting eq.(7) & eq.(8) into eq.(4).

$$(\omega^2-c^2k^2)\cancel{E_x} = -i4\pi\omega n_o e\left[\frac{ie}{m_i\omega}\left(1-\frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \frac{ie}{m_e\omega}\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)^{-1}\right]\cancel{E_x}$$

$$(\omega^2-c^2k^2) = \frac{4\pi n_o e^2}{m_i}\left(1-\frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \frac{4\pi n_o e^2}{m_e}\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)^{-1}$$

$$(\omega^2-c^2k^2) = \omega_{pi}^2\left(1-\frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \omega_{pe}^2\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)^{-1}$$

$$(\omega^2-c^2k^2) = \omega_{pi}^2\left(\frac{\omega^2-\omega_{ci}^2}{\omega^2}\right)^{-1} + \omega_{pe}^2\left(\frac{\omega^2-\omega_{ce}^2}{\omega^2}\right)^{-1} \quad \text{—— (9)}$$

Applying low frequency condition,

$$\omega^2 << \omega_{ce}^2 \quad \& \quad \omega^2 << \omega_{ci}^2$$

$$(\omega^2-c^2k^2) = \omega_{pi}^2\left(-\frac{\omega_{ci}^2}{\omega^2}\right)^{-1} + \omega_{pe}^2\left(-\frac{\omega_{ce}^2}{\omega^2}\right)^{-1}$$

$$\omega^2-c^2k^2 = -\left[\frac{\omega_{pi}^2\omega^2}{\omega_{ci}^2} + \omega_{pe}^2\frac{\omega^2}{\omega_{ce}^2}\right]$$

$$\omega^2-c^2k^2 = -\omega^2\left(\omega_{pi}^2\cdot\frac{1}{\omega_{ci}^2} + \omega_{pe}^2\cdot\frac{1}{\omega_{ce}^2}\right)$$

$$\omega^2-c^2k^2 = -\omega^2\left(\frac{4\pi n_o \cancel{e^2}}{\cancel{m_i}}\cdot\frac{m_i^{\cancel{2}}}{\cancel{e^2}B^2} + \frac{4\pi n_o\cancel{e^2}}{\cancel{m_e}}\cdot\frac{m_e^{\cancel{2}}}{\cancel{e^2}B^2}\right]$$

$$\omega^2-c^2k^2 = -\omega^2\left[\frac{4\pi n_o m_i}{B^2} + \frac{4\pi n_o m_e}{B^2}\right]$$

$$\omega^2-c^2k^2 = -\omega^2\frac{4\pi}{B^2}(n_o m_i + n_o m_e) \qquad \because \rho_i = n_o m_i$$

So, $\qquad \because \rho_e = n_o m_e.$

$$\omega^2 - c^2k^2 = -\omega^2 \frac{4\pi \rho}{B_0^2}$$ $\because \rho = \rho_i + \rho_e$
$= n_0 m_i + n_0 m_e.$

$$\omega^2 + \omega^2 \frac{4\pi\rho}{B_0^2} = c^2k^2$$

$$\omega^2\left(1 + \frac{4\pi\rho}{B_0^2}\right) = c^2k^2 \quad \text{--- 10)}$$

$$V_A^2 = \left(\frac{4\pi\rho}{B_0^2 c^2}\right)^{-1} \text{ is Alfven velocity} \Rightarrow V_A^2 = \frac{B_0^2 c^2}{4\pi\rho}$$

Multiplying and dividing eq. (10) by c^2.

$$\omega^2\left(1 + \frac{4\pi\rho c^2}{B_0^2 c^2}\right) = c^2k^2.$$

$$\omega^2\left(1 + \frac{c^2}{V_A^2}\right) = c^2k^2.$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + \frac{c^2}{V_A^2}}$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{\frac{V_A^2 + c^2}{V_A^2}}$$

$$\frac{\omega^2}{k^2} = \frac{c^2 V_A^2}{V_A^2 + c^2}$$ $\because V_A << C$, so we can write $V_A^2 + c^2 = c^2$.

$$\frac{\omega^2}{k^2} = \frac{c^2 V_A^2}{c^2} = V_A^2$$

It is the dispersion relation of Alfven wave.

$$\frac{\omega^2}{k^2} = V_A^2 = \frac{B_0^2 c^2}{4\pi\rho_0}$$

On the L.H.S, we have wave character & on R.H.S we have particle character. It means wave & particles are stuck together. Particle & wave are move together. Example: Guitar.